LII. *A Demonstration of the Second Rule in the Essay towards the Solution of a Problem in the Doctrine of Chances, published in the Philosophical Transactions,* Vol. LIII. *Communicated by the Rev.* Mr. Richard Price, *in a Letter to* Mr. John Canton, M. A. F. R. S.

Dear Sir,   Nov. 26, 1764.

Read Dec. 6, 1764.   I Send you the following *Supplement to the Essay on a Problem in the Doctrine of Chances,* hoping that you may not think it improper to be communicated to the Royal Society. I should not have troubled you again in this way had I not found that some additions to my former papers were necessary in order to explain some passages in them, and particularly what is hinted in the note at the end of the Appendix. " I have first given the deduction of Mr. Bayes's se- " cond rule chiefly in his own words; and then " added, as briefly as possible, the demonstrations of " several propositions, which seem to improve con- " siderably the solution of the problem, and to throw " light on the nature of the curve by the quadrature " of which this solution is obtained." Perhaps, there is no reason for being very anxious about pro- ceeding to further improvements. It would, how- ever, be very agreeable to me to see a yet easier and nearer approximation to the value of the two series's in the first rule : but this I must leave abler persons to seek, chusing now entirely to drop this subject.

The

The folution of the problem enquired after in the papers I have fent you has, I think, been hitherto a *defideratum* in philofophy of fome confequence. To this we are now in a great meafure helped by the abilities and fkill of our late worthy friend; and thus are furnifhed with a neceffary guide in determining the nature and proportions of unknown caufes from their effects, and an effectual guard againft one great danger to which philofophers are fubject; I mean, the danger of founding conclufions on an infufficient induction, and of receiving juft conclufions with more affurance than the number of experiments will warrant. I am, under a fenfe of the value of your friendfhip, heartily yours,

Richard Price.
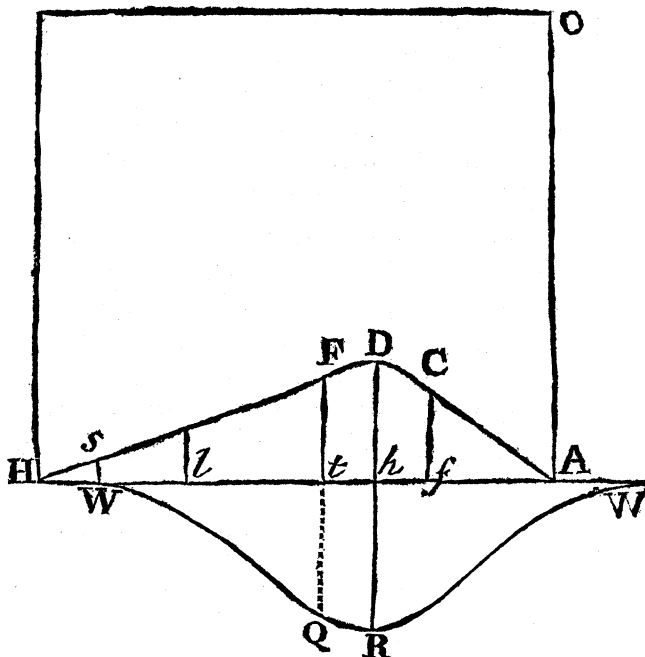
A RT. 1. If the curve A D H be divided into two parts by the ordinate D b making A b to H b as p is to q; then taking $a = \frac{p}{n}$ and $b = \frac{q}{n}$ the ratio of the Area A D b to H O will be $\overline{a \times b^q}^{p+1} \times$

$$\frac{}{1 + \frac{q}{p+2} \times \frac{p}{q} + \frac{q \times \overline{q-1} \times p^2}{p+2 \times \overline{p+3} \times q^2} + \frac{q \times \overline{q-1} \times \overline{q-2} \times p^3}{p+2 \times \overline{p+3} \times \overline{p+4} \times q^3}}$$

$+$ &c. For the feries $\dfrac{x^{p+1} r^q}{p+1} + \dfrac{q}{p+1} \times \dfrac{x^{p+2} \times r^{q-1}}{p+2}$

$+$ &c. in Prop: 10. Art. 2. of the Effay, which ex-preffes the ratio of A C f to H O, becomes this feries when $x = a = \frac{p}{n}$, $b = r = \frac{q}{n}$; that is when

C f

C f has moved till it coincides with D b and A C f becomes A D b. In like manner, from Art. 3. in the Effay, it appears that the ratio of H D b to H O is

$$\frac{\overset{p}{a}\overset{q+1}{b}}{q+1} \times \overline{1 + \frac{p}{q+2} \times \frac{q}{p} + \frac{p}{q+2} \times \frac{p-1}{q+3} \times \frac{q^2}{p^2}} + \&c.$$

From hence it follows that the ratio of the difference between A D b and H D b to H O is $\dfrac{\overset{p}{a}\overset{q}{b}}{n}$ multiplied by

the difference between the feries $\dfrac{p}{p+1} + \dfrac{q}{p+1}$

$\times \dfrac{p^2}{pq+2q} + \dfrac{q \times \overline{q-1} \times p^3}{\overline{p+1} \times \overline{p+2} \times \overline{pq^2+3q^2}} + \&c.$

and the feries $\dfrac{q}{q+1} + \dfrac{p \times q^2}{\overline{q+1} \times \overline{pq+2p}} +$

$\dfrac{p \times \overline{p-1} \times q^3}{\overline{q+1} \times \overline{q+2} \times \overline{p^2q+3p^2}} + \&c.$

the former feries being to be fubftracted from the latter, if H D b is greater than A D b, and *vice verfa*.

2. The ratio of any term in the former of the two foregoing feries to that which next but one follows the correfpondent term in the latter is $\dfrac{pq+p}{p \times q} \times$

$\dfrac{pq+2p}{p \times q} \times \dfrac{p \times q}{qp+q} \times \dfrac{pq+3p}{pq-q} \times \dfrac{pq}{pq+2q} \times \dfrac{pq+4p}{pq-2q}$

$\times \dfrac{pq-p}{pq+3q} \times \dfrac{pq+5p}{pq-3q} \times \dfrac{pq-2p}{pq+4q} \times \dfrac{pq+6p}{pq-4q}$ $\&c.$ taking twice as many terms and four over as there are units in the number which exprefles the diftance of the term in the former feries from its firft term; which

ratio

ratio if q be greater than p, it is evident muſt be greater than the ratio of equality. Wherefore, if from the ſecond ſeries you ſubſtract the two firſt terms which together are leſs than two, the remainder is leſs than the former ſeries; and of conſequence, the former ſeries ſubſtracted from the latter cannot leave a remainder ſo great as two. And therefore in this caſe, that is, when q is greater than p, by the preceding article, the ratio of $HDb - ADb$ to HO cannot be ſo great as $\dfrac{2ab^{pq}}{n}$.

3. The curve ADH being as before divided into two parts ADb and HDb, let the ordinates Cf and Ft be placed on each ſide of Db and at the ſame diſtance from it, and let z be the ratio of bf or bt to AH. Then if y, x and r be reſpectively the ratios of Cf, Af and Hf to AH, by the nature of the curve $y = x^p r^q$. But becauſe the ratio of Ab to AH is a, and that of bf to AH is z, the ratio of $Ab - bf (= Af)$ to AH is $a - z$. Wherefore $a - z = x$. And in like manner $b + z = r$. But $y = x^p r^q$, and y is the ratio of Cf to AH. Wherefore the ratio of Cf to AH is $\overline{a - z}|^p \times \overline{b + z}|^q$. And in like manner the ratio of Ft to AH is $\overline{a + z}|^p \times \overline{b - z}|^q$. And conſequently Cf is to Ft as $\overline{a - z}|^p \times \overline{b + z}|^q$ is to $\overline{a + z}|^p \times \overline{b - z}|^q$.

4. If q is greater than p, $\overline{a + z}|^p \times \overline{b - z}|^q$ is greater than $\overline{a - z}|^p \times \overline{b + z}|^q$, and the ratio between them increaſes as z increaſes. For the hyperbolic logarithm of

of that ratio taken as ufual, and then inftead of p and q putting $n\,a$ and $n\,b$ becaufe $a = \frac{p}{n}$ and $b = \frac{q}{n}$ (Vid. Art. 1.) you will find to be $2\,n$ multiplied by the feries $\frac{b^2 - a^2}{3\,b^2\,a^2} \times z^3 + \frac{b^4 - a^4}{5\,b^4\,a^4} \times z^5 + \frac{b^6 - a^6}{7\,b^6\,a^6} \times z^7$ &c. which logarithm when q is greater than p, and therefore b greater than a has all its terms pofitive, and fo much the greater as z is greater; and therefore it is the logarithm of a ratio greater than that of equality, and which increafes as z increafes.

5. By Art. 3. F t is to C f as $\overline{a+z}|^p \times \overline{b-z}|^q$ is to $\overline{a-z}|^p \times \overline{b+z}|^q$. And by Art. 4. $\overline{a+z}|^p \times \overline{b-z}|^q$ is greater than $\overline{a-z}|^p \times \overline{b+z}|^q$, and the ratio between them increafes as z increafes, if q is greater than p. Wherefore, upon this fuppofition, alfo F t is greater than C f, and the ratio between them increafes as z or $b\,t$ and $b\,f$ increafes, and confequently this will be true alfo concerning the areas defcribed by them as their equal abfciffes $b\,t$ and $b\,f$ increafe. Wherefore, when q is greater than p, D $b\,t$ F is greater than D $b\,f$ C, and the ratio between them increafes as $b\,f = b\,t$ increafes.

6. Becaufe A b is to H b as p is to q, when q is greater than p, H b is greater than A b. In H b therefore taking $b\,l$ equal to A b, by the preceding Art. the part of the figure H D b which infifts upon $b\,l$ will be greater than A D b, and the ratio of that part of H D b to A D b will be greater than the ratio of D $b\,t$ F to D $b\,f$ C. Confequently, much more (q being greater than p) the whole figure H D b is

greater

greater than A D h, and the ratio of H D h to A D h
is greater than that of D $h\,t$ F to D $h\,f$ C.

7. When q is greater than p, $\overline{1 - \dfrac{n^2 z^2}{p\,q}}\Big|^{\frac{n}{2}}$ is greater

than $\overline{1 - \dfrac{n\,z}{p}}\Big|^{p} \times \overline{1 + \dfrac{n\,z}{q}}\Big|^{q}$ and lefs than $\overline{1 - \dfrac{n\,z}{q}}\Big|^{q}$

$\times \overline{1 + \dfrac{n\,z}{p}}\Big|^{p}$. For the fluxion of $\overline{1 - \dfrac{n^2 z^2}{p\,q}}\Big|^{\frac{n}{2}}$ is

$-\dfrac{n^3 z \dot{z}}{p\,q} \times \overline{1 - \dfrac{n^2 z^2}{p\,q}}\Big|^{\frac{n}{2} - 1}$ and the fluxion of

$\overline{1 - \dfrac{n\,z}{p}}\Big|^{p} \times \overline{1 + \dfrac{n\,z}{q}}\Big|^{q}$ (becaufe $p + q = n$) is

$-\dfrac{n^3 z \dot{z}}{p\,q} \times \overline{1 - \dfrac{n\,z}{p}}\Big|^{p-1} \times \overline{1 + \dfrac{n\,z}{q}}\Big|^{q-1}$. Wherefore

$\overline{1 - \dfrac{n^2 z^2}{p\,q}}\Big|^{\frac{n}{2}}$ is to $\overline{1 - \dfrac{n\,z}{p}}\Big|^{p} \times \overline{1 + \dfrac{n\,z}{q}}\Big|^{q}$ as the flux-

ion of the former multiplied by $\overline{1 - \dfrac{n^2 z^2}{p\,q}}\Big|$ to the

fluxion of the latter multiplied by $(\overline{1 - \dfrac{n\,z}{p}} \times$

$\overline{1 + \dfrac{n\,z}{q}}$ or) $1 - \dfrac{n\,z}{p} + \dfrac{n\,z}{q} - \dfrac{n^2 z^2}{p\,q}$. From which

analogy, becaufe q is greater than p, it is plain that

$\overline{1 - \dfrac{n\,z}{p}}\Big|^{p} \times \overline{1 + \dfrac{n\,z}{q}}\Big|^{q}$ varies at a greater rate in refpect

of its own magnitude than $\overline{1 - \dfrac{n^2 z^2}{p\,q}}\Big|^{\frac{n}{2}}$ does. And,

becaufe their fluxions as found out before have a ne-
gative fign before them, they both decreafe as z in-
creafes;

creafes; confequently, if they are ever equal, as z increafes the latter muft be the largeft. But when z = o they are each equal to 1. In like manner the other part of this article appears. And hence, fince $a = \frac{p}{n}$ and $b = \frac{q}{n}$, it is manifeft that $a^p b^q$ $\times \overline{1 - \frac{n^2 z^2}{p q}}\Big|^{\frac{n}{2}}$ is greater than $\overline{a - z}\Big|^p \times \overline{b + z}\Big|^q$ and lefs than $\overline{a + z}\Big|^p \times \overline{b - z}\Big|^q$, when q is greater than p.

8. Suppofe now further that the curve R Q W be defcribed meeting the lines D h, F t, $h t$ produced in R, Q, W, in fuch manner that F t, which is to C f as $\overline{a + z}\Big|^p \times \overline{b - z}\Big|^q$ to $\overline{a - z}\Big|^p \times \overline{b + z}\Big|^q$ (Vid. Art. 3.) fhall be to Q t as $\overline{a + z}\Big|^p \times \overline{b - z}\Big|^q$ to $a^p b^q \times$ $1 - \overline{\frac{n^2 z^2}{p q}}\Big|^{\frac{n}{2}}$ wherever the points t and f fall at equal diftances from h. And it is manifeft by the foregoing Art. that Q t muft always be greater than C f, and lefs than F t. And of confequence the fame muft be true concerning the areas defcribed by their motion while their equal abfciffes increafe. Wherefore R h t Q is greater than D $h f$ C, and lefs than D h t F.

9. Since F t is to Q t as $\overline{a + z}\Big|^p \times \overline{b - z}\Big|^q$ to $a^p b^q \times \overline{1 - \frac{n^2 z^2}{p q}}\Big|^{\frac{n}{2}}$; and $\overline{a + z}\Big|^p \times \overline{b - z}\Big|^q$ (by

Art..

Art. 3.) expreſſes the ratio of F t to A H; the ratio

of Q t to A H muſt be $a^p b^q \times \overline{1 - \frac{n^2 z^2}{pq}}^{\frac{n}{2}}$, and as

has been all along ſuppoſed z is the ratio of $h t$ to A H. Wherefore, by ſquaring the curve R $h t$ Q, it will appear that the ratio of R $h t$ Q to H O is

$$a^p b^q \times \overline{z - \frac{n^3 z^3}{2 \cdot 3\, pq} + \frac{n-2}{4} \times \frac{n^5 2^5}{2 \cdot 5\, p^2 q^2} - \frac{n-2}{4} \times}$$

$$\overline{\frac{n-4}{6} \times \frac{n^7 z^7}{2 \cdot 7\, p^3 q^3} + \&c.} \text{ which } \left(\text{if } m^2 = \frac{n^3}{2\, p\, q} \right) \text{ is}$$

$$a^p b^q \times \frac{\sqrt{2pq}}{n \sqrt{\frac{=}{n}}} \times \overline{mz - \frac{m^3 z^3}{3} + \frac{n-2}{2n} \times \frac{m^5 z^5}{5} - \frac{n-2}{2n}}$$

$$\overline{\times \frac{n-4}{3n} \times \frac{m^7 z^7}{7} + \frac{n-2}{2n} \times \frac{n-4}{3n} \times \frac{n-6}{4n} \times \frac{m^9 z^9}{9}}$$

$- \&c.$ Which laſt ſeries when $\frac{n^2 z^2}{pq} = 1$, and conſequently the ordinate Q t vaniſhes, becomes B $-$

$$\frac{B^3}{3} + \frac{B^2 - 1}{{}^2 B^2} \times \frac{B^5}{5} - \frac{B^2 - 1}{{}^2 B^2} \times \frac{B^2 - 2}{{}^3 B^2} \times \frac{B^7}{7} + \&c.$$

taking B$^2 = \frac{n}{2}$.

10. If B$^2 = \frac{n}{2}$ the ratio of the whole figure

R Q W h to H O is $\frac{\sqrt{2pq}}{n \sqrt{\frac{=}{n}}} \times a^p b^q \times \overline{B - \frac{B^3}{3} +}$

$\frac{B^2 - 1}{{}^2 B^2} \times \frac{B^5}{5} - \&c.$ Now, (by Prop. 10. Art. 4. of

the Eſſay) the ratio of A C F H to H O is $\frac{1}{n+1}$

X

$\times \frac{1}{E}$, E being the coefficient of that term of the binomial $\overline{a+b}^n$ expanded in which occurs $a^p b^q$: Wherefore, the ratio of R Q W b to A C F H is

$$\frac{n+1}{n} \times \frac{\sqrt{2pq}}{\sqrt{n}} \times E \, a^p b^q \times \overline{B - \frac{B^3}{3} + \frac{B^2-1}{2B^2} \times \frac{B^5}{5} \mathcal{E}c.}$$

Put G now for the coefficient of the middle term of the same binomial, and if $p = q = \frac{n}{2}$, E = G, $a = \frac{1}{2} = b$ the area R Q W b is equal to half A C F H; for then Q t, F t, C f are all equal, and confequently the areas R Q W b, H D b and A D b. Wherefore, the

feries $B - \frac{B^3}{3} + \mathcal{E}c.$ is equal to $\frac{\sqrt{2n}}{n+1} \times \frac{2^{n-1}}{G}$. But the feries $B - \frac{B^3}{3} + \mathcal{E}c.\left(\text{becaufe } B^2 = \frac{n}{2}\right)$ does not alter whatever p and q are, whilft their fum n remains the fame. Wherefore, in all cafes, the ratio of R Q W b to A C F H is $\frac{\sqrt{pq}}{n} \times \frac{E \, a^p b^q}{G} \times 2^n$.

11. By Prop. 10. Art. 4. of the Effay, the ratio of A C F H to HO* is $\frac{1}{n+1} \times \frac{1}{E}$; and by Art. 9. the ratio of R $b t$ Q to H O is $a^p b^q \times \frac{\sqrt{2pq}}{n\sqrt{n}} \times \overline{m \, z - \frac{m^3 z^3}{3} + \frac{n-2}{2n} \times \frac{m^5 z^5}{5}} \mathcal{E}c.$ Wherefore, the ratio of

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* It is hoped that the imperfection of the figure all along referred to will be excufed. The lines R b and D b fhould appear equal; and it will be found prefently, that the curve line A C D F H fhould have been drawn from F and C convex towards A H.

$R\,b\,t\,Q$ to $ACFH$ is $\dfrac{n+1}{n} \times \dfrac{\sqrt{2\,p\,q}}{\sqrt{n}} \times E\overset{p}{a}\overset{q}{b} \times \overline{m\,z - \dfrac{\overline{m^3\,z^3}}{3}}$

$+ \dfrac{n-2}{2\,n} \times \dfrac{m^5\,z^5}{5} - \dfrac{n-2}{2\,n} \times \dfrac{n-4}{3\,n} \times \dfrac{m^7\,z^{i7}}{7} + \&c.$

Likewife, by Art. 10. the ratio of $R\,Q\,W\,b$ to

$ACFH$ is $\dfrac{\sqrt{p\,q}}{n} \times \dfrac{E\overset{p}{a}\overset{q}{b}}{G} \times \overset{n}{2}.$ Wherefore the ra-

tio of $R\,b\,t\,Q$ to $R\,Q\,W\,b$ is $\dfrac{n+1}{\sqrt{n}} \times \dfrac{\sqrt{2}}{2\,n} \times G$

$\times \overline{m\,z - \dfrac{m^3\,z^3}{3}} + \&c.$

12. By Art. 2. 6. When q is greater than p, the ratio of $HD\,b - AD\,b$ to HO is lefs than $\dfrac{2\,\overset{p}{a}\overset{q}{b}}{n}$. And by Prop. 10. Art. 4. of the Effay, the ratio of $ACFH$ or $HD\,b + AD\,b$ to HO is $\dfrac{1}{n+1} \times \dfrac{1}{E}.$ Wherefore, the fum of thefe two ratios, or the ratio of $2\,HD\,b$ to HO, is lefs than $\dfrac{1}{n+1}$

$\times \dfrac{1}{E} + \dfrac{2\,\overset{p}{a}\overset{q}{b}}{n};$ and the difference between them, or the ratio of $2\,AD\,b$ to HO is greater than $\dfrac{1}{n+1} \times \dfrac{1}{E} - \dfrac{2\,\overset{p}{a}\overset{q}{b}}{n}.$ Wherefore, the ratio of $2\,HD\,b$ to $2\,AD\,b$, or that of $HD\,b$ to $AD\,b$, is lefs than

that of $\dfrac{1}{n+1} \times \dfrac{1}{E} + \dfrac{2\,\overset{p}{a}\overset{q}{b}}{n}$ to $\dfrac{1}{n+1} \times \dfrac{1}{E} - \dfrac{2\,\overset{p}{a}\overset{q}{b}}{n}$, which

is equal to the ratio of $1 \times 2\,E\,\overset{p}{a}\overset{q}{b} + \dfrac{2\,E\overset{p}{a}\overset{q}{b}}{n}$ to 1.

$— 2$

— 2 E $a^p b^q$ — $\frac{2 \, \mathrm{E} \, a^p b^q}{n}$. But the ratio of H D h to A D h, by Art. 6. is greater than the ratio of D $h t$ F to D $h f$ C, when q is greater than p. Wherefore, much more when q is greater than p, the ratio of D $h t$ F to D $h f$ C will be lefs than that of $1 +$ 2 E $a^p b^q + \frac{2 \, \mathrm{E} \, a^p b^q}{n}$ to $1 -$ 2 E $a^p b^q - \frac{2 \, \mathrm{E} \, a^p b^q}{n}$. And becaufe, by Art. 8. R $h t$ Q is a mean between D $h t$ F and D $h f$ C, the ratio of D $h t$ F to R $h t$ Q will be lefs than that of $1 + $ 2 E $a^p b^q + \frac{2 \, \mathrm{E} \, a^p b^q}{n}$ to

$1 -$ 2 E $a^p b^q - \frac{2 \, \mathrm{E} \, a^p b^q}{n}$. And the ratio of D $h f$ C to R $h t$ Q will be greater than that of $1 -$ 2 E $a^p b^q$ $- \frac{2 \, \mathrm{E} \, a^p b^q}{n}$ to $1 +$ 2 E $a^p b^q + \frac{2 \, \mathrm{E} \, a^p b^q}{n}$.

R U L E II.

If nothing is known of an event but that it has happened p times and failed q in $p + q$ or n trials, and q be greater than p; and from hence I judge that the probability of its happening in a fingle trial lies between $\frac{p}{n}$ and $\frac{p}{n} + z$, (if $m^2 = \frac{n^3}{2 p q}$, $a = \frac{p}{n}$, $b = \frac{q}{n}$, E the coefficient of the term in which occurs $a^p b^q$ when $\overline{a + b}|^n$ is expanded, and $\Sigma =$

$\frac{n+1}{n} \times \frac{\sqrt{2 p q}}{\sqrt{n}} \times \mathrm{E} \, a^p b^q \times \overline{m z - \frac{m^3 z^3}{3} + \frac{n-2}{2 \, n} \times \frac{m^5 z^5}{5}}$

$n-2$

$\overline{\frac{n-2}{2n}} \times \overline{\frac{n-4}{3n}} \times \overline{\frac{m^7 z^7}{7}} + \&c.$) my chance to be in the right is greater than Σ, and lefs than $\Sigma \times$

$$\frac{1 + 2\,E\,a^p b^q + \frac{2\,E\,a^p b^q}{n}}{1 - 2\,E\,a^p b^q - \frac{2\,E\,a^p b^q}{n}}.$$ For by Art. 11. com-

pared with the value of Σ here fet down, the ratio of R b t Q to A C F H is Σ. But by Art. 8. D b t F is greater than R b t Q, and by Art. 12. the ratio of D b t F to R b t Q is lefs than that of $1 + 2\,E\,a^p b^q$

$+ \frac{2\,E\,a^p b^q}{n}$ to $1 - 2\,E\,a^p b^q - \frac{2\,E\,a^p b^q}{n}$. From whence it is plain that the ratio of D b t F to A C F H is

greater than Σ, and lefs than $\Sigma \times \dfrac{1 + 2\,E\,a^p b^q + \frac{2\,E\,a^p b^q}{n}}{1 - 2\,E\,a^p b^q - \frac{2\,E\,a^p b^q}{n}}$.

But, as appears from the 10$^{\text{th}}$ Propofition in the Effay, the chance that the probability of the event lies between $\frac{p}{n}$ and $\frac{p}{n} + z$ (that is, between the ratio of A b to A H, and that of A t to A H) is the ratio of D b t F to A C F H. Wherefore, the chance I am right in my guefs is greater than Σ and lefs than $\Sigma \times$

$$\frac{1 + 2\,E\,a^p b^q + \frac{2\,E\,a^p b^q}{n}}{1 - 2\,E\,a^p b^q - \frac{2\,E\,a^p b^q}{n}}.$$

In

In like manner, 2dly, the fame things fuppofed, if I judge that the probability of the event lies between $\frac{p}{n}$ and $\frac{p}{n} - z$, my chance to be right is lefs than Σ,

and greater than $\Sigma \times \dfrac{1 - 2\,E\,a^p\,b^q - \dfrac{2\,E\,a^p\,b^q}{n}}{1 + 2\,E\,a^p\,b^q + \dfrac{2\,E\,a^p\,b^q}{n}}$. This

is manifeft as the other cafe, becaufe D $b f$ C is lefs than R $b t$ Q, but the ratio between them is greater

than that of $1 - 2\,E\,a^p b^q - \dfrac{2\,E\,a^p b^q}{n}$ to $1 + 2\,E\,a^p b^q +$

$\dfrac{2\,E\,a^p b^q}{n}$.

3dly, If, other things fuppofed as before, p is greater than q, and I judge the probability of the event lies between $\frac{p}{n}$ and $\frac{p}{n} + z$, my chance to be right is lefs than Σ, and greater than $\Sigma \times$

$\dfrac{1 - 2\,E\,a^p\,b^q - \dfrac{2\,E\,a^p\,b^q}{n}}{1 + 2\,E\,a^p\,b^q + \dfrac{2\,E\,a^p\,b^q}{n}}$. But if I judge it lies be-

tween $\frac{p}{n}$ and $\frac{p}{n} - z$, my chance is greater than Σ, and

lefs than $\Sigma \times \dfrac{1 + 2\,E\,a^p\,b^q + \dfrac{2\,E\,a^p\,b^q}{n}}{1 - 2\,E\,a^p\,b^q - \dfrac{2\,E\,a^p\,b^q}{n}}$. And if $p = q$,

which

which ever of thefe ways I guefs, my chance is Σ exactly. This may be proved in the fame manner with the foregoing cafes, where q is greater than p, or may be proved from them by confidering the happening and failing of an event, as the fame with the failing and happening of its contrary.

4thly, Other things fuppofed the fame, whether q be greater or lefs than p, and I judge that the probability of the event lies between $\frac{p}{n} + z$ and $\frac{p}{n} - z$, my chance is greater than $\dfrac{2\,\Sigma}{1 + 2\,\mathrm{E}\,a^p\,b^q + \dfrac{2\,\mathrm{E}\,a^p\,b^q}{n}}$, and lefs than $\dfrac{2\,\Sigma}{1 - 2\,\mathrm{E}\,a^p\,b^q - \dfrac{2\,\mathrm{E}\,a^p\,b^q}{n}}$. This is an evident corollary from the cafes already determined. And here, if $p = q$, my chance is $2\,\Sigma$ exactly.

Thus far I have tranfcribed Mr. Bayes.

It appears, from the Appendix to the Effay, that the rule here demonftrated, though of great ufe, does not give the required chance within limits fufficiently narrow. It is therefore neceffary to look out for a contraction of thefe limits; and this, I think, we fhall difcover by the help of the following deductions; which, for the fake of greater dinftinctnefs, I fhall give as a continuation of the foregoing Articles.

13. The

13. The ratio of the fluxion of $\overline{1 - \frac{n^2 z^2}{pq}}\Big|^{\frac{n}{2}}$ to the

fluxion of $\overline{1 + \frac{nz}{p}}\Big|^{p} \times \overline{1 - \frac{nz}{q}}\Big|^{q}$ is $\dfrac{\overline{1 - \frac{n^2 z^2}{pq}}\Big|^{\frac{n}{2}-1}}{\overline{1 + \frac{nz}{p}}\Big|^{p-1} \times \overline{1 - \frac{nz}{q}}\Big|^{q-1}}$;

and the ratio of the fluxion of $\overline{1 - \frac{nz}{p}}\Big|^{p} \times \overline{1 + \frac{nz}{q}}\Big|^{q}$

to the fluxion of $\overline{1 - \frac{n^2 z^2}{pq}}\Big|^{\frac{n}{2}}$ is $\dfrac{\overline{1 - \frac{nz}{p}}\Big|^{p-1} \times \overline{1 + \frac{nz}{q}}\Big|^{q-1}}{\overline{1 - \frac{n^2 z^2}{pq}}\Big|^{\frac{n}{2}-1}}$.

This will immediately appear from Art. 7.

14. While z is increasing from nothing till $\frac{n^2 z^2}{pq}$ becomes equal to unity, these two ratios are at first greater than the ratio of equality, and increase as z increases, till they come to a *maximum*. Afterwards they decrease untill they become first equal to the ratio of equality, and then less. This is proved by finding the hyperbolic logarithms of these ratios. The hyperbolic logarithm

of the first is the series $\overline{\frac{q-1}{q} - \frac{p-1}{p}} \times nz +$

$\overline{\frac{q-1}{q^2} + \frac{p-1}{p^2} - \frac{n-2}{pq}} \times \frac{n^2 z^2}{2} + \overline{\frac{q-1}{q^3} - \frac{p-1}{p^3}} \times$

$\frac{n^3 z^3}{3} + \overline{\frac{q-1}{q^4} + \frac{p-1}{p^4} - \frac{n-2}{p^2 q^2}} \times \frac{n^4 z^4}{4} + \overline{\frac{q-1}{q^5}}$

4

$$- \frac{p-1}{p^5} \times \frac{n^5 z^5}{5} + \frac{q-1}{q^6} + \frac{p-1}{p^6} - \frac{n-2}{p^3 q^3} \times \frac{n^6 z^6}{6}$$

$+$ &c. The hyperbolic logarithm of the second ratio is the series $\frac{q-1}{q} - \frac{p-1}{p} \times n z - \frac{q-1}{q^2} +$

$$\frac{p-1}{p^2} - \frac{n-2}{p q} \times \frac{n^2 z^2}{2} + \frac{q-1}{q^3} - \frac{p-1}{p^3} \times \frac{n^3 z^3}{3} -$$

$\frac{q-1}{q^4} + \frac{p-1}{p^4} - \frac{n-2}{p^2 q^2} \times \frac{n^4 z^4}{4} +$ &c. It will ap-
pear from examining thefe two feriefes (q all along
fuppofed greater than p) that while z is fmall the
value of each of them is pofitive, and increafes as z
increafes till it becomes a *maximum*, after which it
decreafes till it becomes nothing, and after that ne-
gative ; which demonftrates this article.

15. The former of the two ratios in Art. 13. (q be-
ing greater than p) is at firft, while z is increafing
from nothing, lefs than the fecond ratio; and does
not become equal to it, till fome time after both
ratios have been the greateft poffible.

Upon confidering the two feriefes in the laft Art. it
will appear that the firft term of the firft feries is always
pofitive, the fecond negative, the third alfo negative,
after which the terms become alternately pofitive and
negative. On the other hand, it will appear that in the
fecond feries the two firft terms are always pofitive,
and all that follow negative. But as the feriefes con-
verge very faft when z is fmall, the fecond term
being negative in the firft feries and pofitive in the
fecond, has a greater effect in making the firft feries
lefs than the fecond, than can be compenfated for by
the terms being afterwards alternately negative and

I   pofitive

positive in the one, and all negative in the other. It will further appear from considering these seriefes, that the first must continue less than the second 'till z becomes so large as to make the fourth term equal to the second, in which circumstances the two seriefes are nearly equal. Afterwards, as z goes on to increase, the value of both lessens continually; but the second now decreasing fastest, as before it increased fastest, becomes first nothing. After which, the other series becomes nothing; and after that both remain negative. From hence it is easy to infer this Article.

16. What has been now shewn of the ratio of the fluxion of $\overline{1 - \frac{n^2 z^2}{p\,q}}\Big|^{\frac{n}{2}}$ to the fluxion of $\overline{1 + \frac{n\,z}{p}}\Big|^{p} \times$ $\overline{1 - \frac{n\,z}{q}}\Big|^{q}$ compared with the ratio of the fluxion of $\overline{1 - \frac{n\,z}{p}}\Big|^{p} \times \overline{1 + \frac{n\,z}{q}}\Big|^{q}$ to the fluxion of $\overline{1 - \frac{n^2 z^2}{p\,q}}\Big|^{\frac{n}{2}}$ is also true of the ratio of the fluxion of $a^p\, b^q \times$ $\overline{1 - \frac{n^2 z^2}{p\,q}}\Big|^{\frac{n}{2}}$ (or $Q\,t$ in the figure) to the fluxion of $\overline{a + z}\Big|^p \times \overline{b - z}\Big|^q$ (or $F\,t$) compared with the ratio of the fluxion of $\overline{a - z}\Big|^p \times \overline{b + z}\Big|^q$ (or $C\,f$) to the fluxion of $a^p\, b^q \times \overline{1 - \frac{n^2 z^2}{p\,q}}\Big|^{\frac{n}{2}}$ or $Q\,t$; the latter quantities being only the former multiplied by the common and permanent quantity $a^p\, b^q$. It appears, therefore, that if we conceive $F\,t$, $Q\,t$, $C\,f$ (Vid.

Fig.) to move with equal and uniform velocities, from D h and R h along A H, in order to generate the areas H D h, R W h, A D h; C f will at firſt not only decreaſe faſter than Q t, and Q t than F t; but the ratio of the rate at which C f decreaſes to the rate at which Q t decreaſes, will be greater than the ratio of the rate at which Q t decreaſes to the rate at which F t decreaſes. It appears alſo that after ſome time, firſt C f and Q t, and then Q t and F t will come to decreaſe equally; after which, Q t will decreaſe faſter than C f, and F t faſter than Q t.

17. The curves D F H, R Q W, D C A, have each of them a point of contrary flexure; and the value of z, or of the equal abſciſſes at that point, is in all three $\dfrac{\sqrt[p-q]{}}{\sqrt[n^3-n^2]{}}$. This may be found in the common manner, by putting the ſecond fluxions of the ordinates equal to nothing. In the ſingle caſe, when either p or q is equal to unity, one of theſe points vaniſhes, or coincides with A or H.

18. At the points of contrary flexure (q being greater than p) the ratio of the fluxion of Q t to the fluxion of F t is a *maximum*; and the ſame is true of the ratio of the fluxion of C f to the fluxion of Q t. This is found by making the fluxions of the logarithms of theſe ratios, or of

$$\dfrac{\overline{1-\dfrac{n^2 z^2}{p\,q}}^{\tfrac{n}{2}-1}}{\overline{1+\dfrac{n z}{p}}^{p-1}\times\overline{1-\dfrac{n z}{q}}^{q-1}}, \text{ and } \dfrac{\overline{1-\dfrac{n z}{p}}^{p-1}\times\overline{1+\dfrac{n z}{q}}^{q-1}}{\overline{1-\dfrac{n^2 n^2}{p\,q}}^{\tfrac{n}{2}-1}}$$

1   equal

equal to nothing: which will give the value of z equal to $\dfrac{\sqrt{p\,q}}{\sqrt{n^3 - n^2}}$, or the fame with the value of z at the points of contrary flexure.

19. At the points of contrary flexure, the ratio of the fluxion of C f to the fluxion of Q t, is greateſt in compariſon of the ratio of the fluxion of Q t to the fluxion of F t. This is proved by finding the value of z when the fluxion of the former ratio divided by the latter, or of

$$\frac{\overline{1 - \dfrac{n^2\,z^2}{p^2}}^{\,p-1} \times \overline{1 - \dfrac{n^2\,z^2}{q^2}}^{\,q-1}}{\overline{1 - \dfrac{n^2\,z^2}{p\,q}}^{\,n-2}}$$

is nothing, which will ſtill give $z = \dfrac{\sqrt{p\,q}}{\sqrt{n^3 - n^2}}$. The reaſon, therefore, in the nature of the curve, which, as the ordinates flow, keeps at firſt the exceſs of F t above Q t leſs than the exceſs of Q t above C f, operates with its greateſt force at the points of contrary flexure.

20. The greateſt part of the area R Q W h lies between R h, and the ordinate at the point of contrary flexure. By Art. 11 the ratio of R $h t$ Q to R Q W h is

$$\frac{n+1}{\sqrt{n}} \times \frac{\sqrt{2}}{2} \times G \times \overline{m\,z - \frac{m^3\,z^3}{3} + \frac{n-2}{2\,n} \times \frac{m^5\,z^5}{5} -}$$

&c. Subſtitute here $\sqrt{\dfrac{p\,q}{n^3 - n^2}}$ for z, and $\dfrac{2}{\sqrt{n\,K} \times H}$ * for G (K being the ratio of the quadrantal arc to

* This is always the true value of G; but it would be too tedious to give the demonſtration of this here.

radius,

radius, and H the ratio whofe hyperbolic logarithm is $\frac{3}{12\,n} - \frac{15}{360\,n^3} + \frac{63}{1260\,n^5}$ * &c.) and the ratio of R $h\,t$ Q to R Q W h at the point of contrary flex-ure, will be $\frac{n+1}{\sqrt{n}\times\sqrt{n-1}} \times \frac{.797884}{H} \times \overline{1 - \frac{n}{2.\,3.\,\overline{n-1}}} +$

$\frac{n\times\overline{n-2}}{2.5.4.\,\overline{n-1}|^2} - \frac{n.\,\overline{n-2.\,n-4}}{2.3\,7.8.\,\overline{n-1}|^3} + \frac{n.\,\overline{n-2.\,n-4.\,n-6}}{2.3.4.9.16.\,\overline{n-1}|^4} - $ &c.

Now when n is little, the value of this expreffion will be confiderably greater than .6822. It ap-proaches to this continually as n increafes; and when n is large, it may be taken for this exactly. Thus when $n = 6$, this expreffion is equal to .804. When $n = 110$, it is equal to .6903. If we would know the ratio of R $h\,t$ Q to R Q W h, when C f comes to decreafe no fafter in refpect of Q t, than Q t decreafes in refpect of F t; that is, when the excefs of Q t above C f, is greateft in comparifon of the excefs of F t above Q t, it may be found (by putting the fourth term of the feries in the 14th Art. equal to the fecond term, and then finding the value of z) to be about .8426, when n, p, and q are confiderable; and in other cafes greater.

Coroll. 'Tis eafy to gather from hence that in like manner the greateft part of the area A D H lies between the two ordinates at the points of contrary flexure †.

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* Vid. the Second Rule in the Effay, Phil. Tranf. Vol. LIII.

† From this Article may be inferred a method of finding at once, without any labour, whereabouts it is reafonable to judge the probability of an unknown event lies, about which a given number of experiments have been made. For when

21. R $h t$ Q is greater than the arithmetical mean between D $h t$ F and D $h f$ C. This appears from the latter part of Art. 19. for what is there proved of the ordinates muſt hold true of the contemporary areas generated by them. And though beyond the points at which the ratio of the decreaſe of Q t to the decreaſe of F t comes to an equality with the ratio of the decreaſe of Q t to the decreaſe of C f, the exceſs of F t above Q t begins to grow larger than before in reſpect of the exceſs of Q t above C f; yet as it appears from the laſt article, that above five ſixths of the areas R Q W h and A C F H are generated before the ordinates come to theſe points, and as alſo beyond theſe points the ſaid ratios, 'till they become

neither p nor q are very ſmall, or even not leſs than 10, it will be nearly an equal chance, that the probability of the event lies between $\frac{p}{n} + \frac{\sqrt{pq}}{\sqrt{2n^3 - 2n^2}}$ and $\frac{p}{n} - \frac{\sqrt{pq}}{\sqrt{2n^3 - 2n^2}}$. It will be the odds of two to one that it lies between $\frac{p}{n} + \frac{\sqrt{pq}}{\sqrt{n^3 - n^2}}$ and $\frac{p}{n} - \frac{\sqrt{pq}}{\sqrt{n^3 - n^2}}$; and the odds of five to one that it lies between $\frac{p}{n} + \frac{\sqrt{2pq}}{\sqrt{n^3 - n^2}}$ and $\frac{p}{n} - \frac{\sqrt{2pq}}{\sqrt{n^3 - n^2}}$. For inſtance; when $p = 1000, q = 100$, there will be nearly an equal chance, that the probability of the event lies between $\frac{10}{11} + \frac{1}{163}$ and $\frac{10}{11} - \frac{1}{163}$; two to one that it lies between $\frac{10}{11} + \frac{1}{115}$ and $\frac{10}{11} - \frac{1}{115}$; and five to one that it lies between $\frac{10}{11} + \frac{1}{81}$ and $\frac{10}{11} - \frac{1}{81}$.

negative

negative and for fome time afterwards, are but fmall; the effect produced before towards render-ing the excefs of $D\,h\,t\,F$ above $R\,b\,t\,Q$ always lefs than the excefs of $R\,b\,t\,Q$ above $D\,h\,f\,C$, will be fuch as cannot be compenfated for afterwards.

A further proof of this will appear from confider-ing that even when $R\,b\,t\,Q$ is increafed to $R\,Q\,W\,h$, it is but little fhort of the arithmetical mean between $A\,D\,h$ and $H\,D\,h$. For from Art. 11. and 20. it may be inferred that the ratio of the whole area $R\,Q\,W\,h$ to this mean, or to $\dfrac{ACFH}{2}$, is $h\,\times\,H$, which is never far from the ratio of equality, but when both p and q are of any confiderable magni-tude, it is very nearly the ratio of equality. For example; when $n = 110$, $q = 100$, $p = 10$, it is ·9938.

22. The ratio of $D\,h\,t\,F$ to $R\,h\,t\,Q$ is lefs than that of

$$1 + 2\,E\,\overset{p}{a}\,\overset{q}{b} + \frac{2\,E\,\overset{p}{a}\,\overset{q}{b}}{n} \text{ to one.}$$

For by Art. 12. the ratio of $D\,h\,t\,F$ to $D\,h\,f\,C$ is lefs than that of

$$1 + 2\,E\,\overset{p}{a}\,\overset{q}{b} + \frac{2\,E\,\overset{p}{a}\,\overset{q}{b}}{n} \text{ to } 1 - 2\,E\,\overset{p}{a}\,\overset{q}{b} - \frac{2\,E\,\overset{p}{a}\,\overset{q}{b}}{n}.$$

But by the laft Art. $R\,h\,t\,Q$ is greater than the arith-metical mean between $D\,h\,t\,F$ and $D\,h\,f\,C$, and 1 is exactly the arithmetical mean between $1 + 2\,E\,\overset{p}{a}\overset{q}{b}$

$$+\frac{2\,E\,\overset{p}{a}\overset{q}{b}}{n} \text{ and } 1 - 2\,E\,\overset{p}{a}\overset{q}{b} - \frac{2\,E\,\overset{p}{a}\overset{q}{b}}{n}.$$

From whence this Article is plain.

23. The

23. The ratio of $\overline{\mathrm{D}\,h\,t\,\mathrm{F}}$ to $\mathrm{A\,C\,F\,H}$ is greater than Σ, and less than $\Sigma \times \overline{1 + 2\,\mathrm{E}\,\overset{p}{a}\overset{q}{b}} + \dfrac{2\,\mathrm{E}\,\overset{p}{a}\overset{q}{b}}{n}$. For $\mathrm{D}\,h\,t\,\mathrm{F}$ being greater than $\mathrm{R}\,h\,t\,\mathrm{Q}$, the ratio of it to $\mathrm{A\,C\,F\,H}$ must be greater than the ratio of $\mathrm{R}\,h\,t\,\mathrm{Q}$ to $\mathrm{A\,C\,F\,H}$, or greater than Σ. Also; since the ratio of $\mathrm{R}\,h\,t\,\mathrm{Q}$ to $\mathrm{A\,C\,F\,H}$ is equal to Σ; and the ratio of $\mathrm{D}\,h\,t\,\mathrm{F}$ to $\mathrm{R}\,h\,t\,\mathrm{Q}$ is less than the ratio of $1 + 2\,\mathrm{E}\,\overset{p}{a}\overset{q}{b} + \dfrac{2\,\mathrm{E}\,\overset{p}{a}\overset{q}{b}}{n}$ to 1; it follows that the ratio compounded of the ratio of $\mathrm{R}\,h\,t\,\mathrm{Q}$ to $\mathrm{A\,C\,F\,H}$, and of $\mathrm{D}\,ht\,\mathrm{F}$ to $\mathrm{R}\,ht\,\mathrm{Q}$, that is, the ratio of $\overline{\mathrm{D}\,ht\,\mathrm{F}}$ to $\mathrm{A\,C\,F\,H}$ must be less than $\Sigma \times \overline{1 + 2\,\mathrm{E}\,\overset{p}{a}\overset{q}{b}} + \dfrac{2\,\mathrm{E}\,\overset{p}{a}\overset{q}{b}}{n}$.

24. The ratio of $\mathrm{D}\,h\,t\,\mathrm{F} + \mathrm{D}\,h\,f\,\mathrm{C}$ to $\mathrm{A\,C\,F\,H}$ (that is, the chance for being right in judging that the probability of an event perfectly unknown, which has happened p and failed q times in $p + q$ or n trials, lies somewhere between $\dfrac{p}{n} + z$ and $\dfrac{p}{n} - z$) is greater than $\dfrac{2\,\Sigma}{1 + 2\,\mathrm{E}\,\overset{p}{a}\,\overset{q}{b} + \dfrac{2\,\mathrm{E}\,\overset{p}{a}\,\overset{q}{b}}{n}}$, and less than $2\,\Sigma$.

The former part of this Art. has been already proved, Art. 12. The latter part is evident from Art. 21. For $\mathrm{R}\,h\,t\,\mathrm{Q}$ being greater than the arithmetical mean between $\mathrm{D}\,h\,t\,\mathrm{F}$ and $\mathrm{D}\,h\,f\,\mathrm{C}$, $2\,\mathrm{R}\,h\,t\,\mathrm{Q}$ must be greater than $\mathrm{D}\,h\,t\,\mathrm{F} + \mathrm{D}\,h\,f\,\mathrm{C}$; and consequently the

the ratio of 2 R h t Q to A C F H, greater than the ratio of D h t F $+$ D h f C to A C F H *.

It will be eafily feen that this Article improves confiderably the rule given in Art. 12. But we may determine within ftill narrower limits whereabouts the required chance muft lie, as will appear from the following Articles.

25. If c and d ftand for any two fractions, when-ever the fluxion of $c \times F t$ is greater than the fluxion of $d \times C f$ (Vid. fig.) $c \times F t + d \times C f$ will be greater than Q t. For in the fame manner with Art. 6. it will appear that $c \times F t + d \times C f$ is to Q t, as the fluxion of $c \times F t \times \overline{1 + \frac{n z}{q}} \times \overline{1 - \frac{n z}{q}}$ to-gether with the fluxion of $d \times C f \times \overline{1 - \frac{n z}{p}} \times \overline{1 + \frac{n z}{q}}$ to the fluxion of Q $t \times \overline{1 - \frac{n^2 z^2}{p q}}$. Now fince $1 - \frac{n^2 z^2}{p q}$ is the arithmetical mean between $\overline{1 + \frac{n z}{p}} \times \overline{1 - \frac{n z}{q}}$ and $\overline{1 - \frac{n z}{p}} \times \overline{1 + \frac{n z}{q}}$, it is plain, that were the fluxion of $c \times F t$ equal to the fluxion of $d \times C f$, $c \times F t + d \times C f$ would decreafe in refpect of its own magnitude at the fame rate with Q t; and, therefore, fince at firft equal, they would al-ways continue equal. But the fluxion of $c \times F t$ being greater than the fluxion of $d \times C f$ by fuppo-fition, and (fince q greater than) p $\overline{1 + \frac{n z}{p}} \times \overline{1 - \frac{n z}{q}}$,

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* This Art. is true, whether p be greater or lefs than q.

alſo greater than $1 - \frac{n\,z}{p} \times 1 + \frac{n\,z}{q}$, it follows that the fluxion of $c \times \mathrm{F}\,t \times \overline{1 + \frac{n\,z}{p}} \times \overline{1 - \frac{n\,z}{q}}$ added to the fluxion of $d \times \mathrm{C}f \times \overline{1 - \frac{n\,z}{p}} \times \overline{1 + \frac{n\,z}{q}}$ is greater than theſe two fluxions multiplied by $1 - \frac{n^2\,z^2}{p\,q}$; and, therefore, greater, than the fluxion of $\mathrm{Q}\,t \times \overline{1 - \frac{n^2\,z^2}{p\,q}}$; and, therefore, $c \times \mathrm{F}\,t + d \times \mathrm{C}f$ greater than $\mathrm{Q}\,t$.

26. If we ſuppoſe three continued arithmetical means betwen $\mathrm{C}f$ and $\mathrm{F}\,t$ $\left(3\dfrac{\mathrm{C}f + \mathrm{F}\,t}{4}, \ \dfrac{\mathrm{C}f + \mathrm{F}\,t}{2}, \right.$ $\left. \dfrac{3\,\mathrm{F}\,t + \mathrm{C}f}{4} \right)$ $\mathrm{Q}\,t$ will be greater than the ſecond, and leſs than the third, if p is greater than 1. That $\mathrm{Q}\,t$ will be greater than the ſecond has been already proved ; and that it will be leſs than the third, will be an immediate conſequence from the laſt Article, if it can be ſhewn that the fluxion of $\frac{3\,\mathrm{F}\,t}{4}$ is greater than the fluxion of $\frac{\mathrm{C}\,f}{4}$. This will appear in the following manner. The ratio of the fluxion of $\mathrm{C}f$ to the fluxion of $\mathrm{F}\,t$ is by Art. 7. and 14.

$$\frac{\overline{1 - \frac{n\,z}{p}}^{\,p-1} \times \overline{1 + \frac{n\,z}{q}}^{\,q-1}}{\overline{1 + \frac{n\,z}{p}}^{\,p-1} \times \overline{1 - \frac{n\,z}{q}}^{\,q-1}}.$$ The hyperbolic logarithm

T t   of

of this ratio is $\overline{\dfrac{1}{p} - \dfrac{1}{q}} \times 2\,n\,z - \overline{\dfrac{1}{p^2} - \dfrac{1}{p^3} - \dfrac{1}{q^2}} +$

$\overline{\dfrac{1}{q^3}} \times \dfrac{2\,n^3\,z^3}{3} - \overline{\dfrac{1}{p^4} - \dfrac{1}{p^5} - \dfrac{1}{q^4} + \dfrac{1}{q^5}} \times \dfrac{2\,n^5\,z^5}{5},$ &c.

This ratio by Art. 18. is greateft at the point of contrary flexure, or when $z = \dfrac{\sqrt{p\,q}}{\sqrt{n^3 - n^2}}$. Subftitute this

for z in the feries, and it will become $\overline{\dfrac{1}{p} - \dfrac{1}{q}}$

$\times \dfrac{2\sqrt{p\,q}}{\sqrt{n-1}} - \overline{\dfrac{1}{p^2} - \dfrac{1}{p^3} - \dfrac{1}{q^2} + \dfrac{1}{q^3}} \times \dfrac{2\,p^{\frac{3}{2}} \times q^{\frac{3}{2}}}{3 \times \overline{n-1}^{\frac{3}{2}}}$ ⌐,

&c. which, therefore, expreffes the logarithm of the ratio when greateft, and will eafily difcover it in every cafe. 'Tis apparent that the value of this feries is greateft when p is leaft in refpect of q. Suppofe then $p = 2$, and q infinite. In this cafe, the value of the feries will be 1.072, and the number anfwering to this logarithm is not greater than 2.92. The fluxion, therefore, of $C\,f$, when greateft, cannot be three times the contemporary fluxion of $F\,t$; from whence it follows that the fluxion of $\dfrac{3\,F\,t}{4}$ muft be greater than the fluxion of $\dfrac{C\,f}{4}$.

It is eafy to fee how thefe demonftrations are to be varied when q is lefs than p, and how in this cafe fimilar conclufions may be drawn. Or, the fame conclufions will in this cafe immediately appear, by changing p into q and q into p, which will not make any difference in the demonftrations.

In the manner fpecified in this Article we may always find within certain limits how near the value of $Q\,t$ comes to the arithmetical mean between $F\,t$ and $C\,f$, which limits grow narrower and narrower, as

4 p and

p and q are taken larger, or their ratio comes nearer to that of equality, 'till at laft, when p and q are either very great or equal, $Q\,t$ coincides with this mean. Thus, if either p or q is not lefs than 10; that is, in all cafes, where it is not practicable without great difficulty to find the required chance exactly by the firft rule, $Q\,t$ will be greater than the fourth, and lefs than the fifth of feven arithmetical means between $C\,f$ and $F\,t$.

27. The arithmetical means mentioned in the laft Article may be conceived as ordinates defcribing areas at the fame time with $Q\,t$; and what has been proved concerning them is true alfo of the areas defcribed by them compared with $R\,b\,t\,Q$.

28. If either p or q is greater than 1, the true chance that the probability of an unknown event which has happened p times and failed q in $\overline{p+q}$ or n trials, fhould lie fomewhere between $\dfrac{p}{n} + z$ and $\dfrac{p}{n}$ $- z$ is lefs than $2\,\Sigma$, and greaterer than $\Sigma +$

$$\dfrac{\Sigma \times 1 - 2\,E\,a^p b^q - \dfrac{2\,E\,a^p b^q}{n}}{1 + E\,a^p b^q + \dfrac{E\,a^p b^q}{n}}.$$ If either p or q is

greater than 10, this chance is lefs than $2\,\Sigma$, and greater than $\Sigma +$

$$\dfrac{\Sigma \times 1 - 2\,E\,a^p b^q - \dfrac{2\,E\,a^p b^q}{n}}{1 + \frac{1}{2}\,E\,a^p b^q + \dfrac{E\,a^p b^q}{2\,n}}.$$

T t 2   This

This is eafily proved in the fame manner with Art. 12, 23, 24.

That it may appear how far what has been now demonftrated improves the folution of the prefent problem, let us take the fifth cafe mentioned in the Appendix to the Effay, and enquire what reafon there is for judging that the probability of an event concerning which nothing is known, but that it has happened 100 times and failed 1000 times in 1100 trials, lies between $\frac{10}{11} + \frac{1}{110}$ and $\frac{10}{11} - \frac{1}{110}$. The fecond rule as given in Art. 12. informs us, that the chance * for this muft lie between .6512, (or the odds of 186 to 100) and .7700, (or the odds of 334 to 100). But from the laft Art. it will appear that the required chance in this cafe muft lie between 2 Σ, and

$$\Sigma + \Sigma \times \frac{1 - E\, a^p b^q - \dfrac{2\, E\, a^p b^q}{n}}{1 + \frac{1}{10} E\, a^p b^q + \dfrac{E\, a^p b^q}{10\, n}} \, ; \quad \text{or,} \quad \text{between}$$

.6748 and .7057; that is, between the odds of 239 to 100, and 207 to 100.

In all cafes when z is fmall, and alfo whenever the difparity between p and q is not great 2 Σ is almoft exactly the true chance required. And I have reafon to think, that even in all other cafes, 2 Σ gives the

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* In the Appendix, this chance, as difcovered by Mr. Bayes's fecond rule, is given wrong, in confequence of making m^2 equal to $\dfrac{n^3}{p\,q}$, whereas it fhould have been taken equal to $\dfrac{n^3}{2\,p\,q}$ as appears from Article 8.

true

true chance nearer than within the limits now determined. But not to purfue this fubject any further; I fhall only add that the value of $2\ \Sigma$ may be always calculated very nearly, and without great difficulty; for the approximations to the value of $E\ a^p b^q$, and of the feries $m - z\ \dfrac{m^3\ z^3}{3} + \dfrac{n-2}{2 \cdot n} \times \dfrac{m^5\ z^5}{5}$, &c. *

given in the Effay, are fufficiently accurate in all cafes where it is neceffary to ufe them.

* In the expreffion for this laft approximation there is an error of the prefs which fhould be corrected; for the fign before the fourth term fhould be — and not +.